Jordan Ideals In prime Rings and Generalized Derivations

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Abstract : Let Rbe a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. Let F and G be two generalized derivations with associated derivations f and g, respectively, the main result We that if $F(x)x - xG(x) = 0$ for all $x \in J$, then$R$ is commutative and $F = G$ or G is a left multiplier and $F = G + f$.

Index Terms **– Prime Ring, Jordan Ideal, Subring, Homomorphism.**

I. Introduction

Let R will be an associative ring and $Z(R)$ the centre of R. For any$x , y \in R$, the symbol $[x, y]$ and xoy denote the Lie product $xy - yx$and $xy + yx$respectively.We recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $d: R \to R$is called a derivation if $d(xy) = d(x)y + xd(y)$hold for all$x , y \in R$.

In [4] Bresar introduced the definition of a generalized derivation: An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d: R \to R$,called the associated derivation of F, such that $F(xy) = F(x)y + yF(y)$, $\forall x , y \in R$.The notion of generalized derivations covers both the notions of a derivation and of a left multiplier and an additive mapping satisfying $f(xy) = f(x)y$, $\forall x , y \in R$. A ring R is said to be n-torsion free, where $n \neq 0$ is a positive integer, if whenever $na = 0$, with $a \in R$, then $a = 0$. An additive subgroup $J$is said to be a Jordan ideal of Rif $uor \in J$ for all $u \in J$and $r \in R$. All ideal of $R$is a Jordan ideal of R but the Jordan ideal need not be ideal..Anadditive subgroup$U$ of R is said to be a Lie ideal of R if$[u, r] \in U$ for all $u \in U$ and $r \in R$. It is clear that if characteristic of R is 2, then Jordan ideals and Lie ideals of R are coincide.

Several authors have proved commutativity theorems for prime and semi-prime rings admitting derivations of generalized derivations. It is worth mentioning that the investigation in this direction started with Posner in his famous paper [1] (see also the interesting work of Bresar[2]). Recently, in [5], EI-Soufi and Aboubakr proved the following result:

Let R be a 2-torsion free prime ring, J be both a nonzero Jordan ideal and a subring of R, and F be a generalized derivation with associated derivation f. If one of the following properties holds:$(i) F(x)x = xf(x) (ii) F(x^2) = 2F(x)x$

$$(ii) F(x^2) = 2xF(x) (iv) F(x^2) - 2xF(x) = f(x^2) - 2xf(x)$$

For all $x \in J$ then $J \subseteq$ Z(R).

In [5, Example 3.8], they gave an example showing that the above result is not true in general if we assume that J is only a subring of R. In this paper we show that in fact, then condition of J being a subring is redundant. Indeed we prove this fact in a more general context. First, we focus on the generalization of the first assertion which is in fact our main result in this paper. As consequence we get generalization of other assertions.

1. Preliminary results

Let us begin with the following lemmas which will sometimes be used without explicit mention.

Lemma 2.1 ([7], *Lemma* 2.4).If J is a nonzero Jordan ideal of a ring R, then

$$2[R, R] J \subset J \text{ and } 2J [R, R] \subset J.$$

Lemma 2.2([7], *Lemma* 2.6).Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If, for two elements $a, b \in R$, $aJb = (0)$, *then*

$$a = 0 \text{ or } b = 0.$$

Lemma 2.3([7], *Lemma* 2.7).Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R.If $[J, J] = 0$,then R is commutative.

Lemma 2.4([6], *Proof of Lemma* 3).Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. Then,$4 j^2 R \subset J$ *and* $4R j^2 \subset J$, $\forall j \in J$.

Lemma 2.5([6], *Proof of Theorem* 2.12). Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. Then,$4jRj \subset J$, $\forall j \in J$.

We will also make use of the following basic commutator identities:
$[x, yz] = y[x, z] + [x, y]z$ and $[xy, z] = x[y, z] + [x, z]y$

2. Main results

We prove the following particular case of our main theorem.

Lemma 3.1 Let R be a 2-torsion free prime ring and two generalized derivations$F$ and G associated with f and g, respectively. If $F(x)x - xG(x) = 0$ for all $x \in R$, then one of the following condition satisfy:

(1) R is commutative and $F = G$.

(2) G is a left multiplier and $F = G + f$.

Proof. Let R be a 2-torsion free prime ring and two generalized derivations F and G associated with f and g , respectively Assume that

$F(x)x - xG(x) = 0$ for all $x \in R$……………………………………………….(1)

The linearization of (1) gives

$F(x)y + F(y)x - xG(y) + yG(x) = 0$ for all $x, y \in R$………………………..(2)

Replacing y by yx in (2) we get

$F(x)yx + F(yx)x - xG(yx) + yxG(x) = 0$ for all $x, y \in R$

$yf(x)x - xyg(x) - yxG(x) + yG(x)x = 0$ for all $x, y \in R$………………….(3)

Replacing ry by y in (3) we get

$ryf(x)x - xryg(x) - ryxG(x) + ryG(x)x = 0$ for all $r, x, y \in R$……………..(4)

Left multiplying (3) by r we get

$ryf(x)x - rxyg(x) - ryxG(x) + ryG(x)x = 0$ for all $r, x, y \in R$……………….(5)

Subtracting (5) from (4), we get

$rxyg(x) - xryg(x) = 0$ for all $r, x, y \in R$

$(rx - xr) yg(x) = 0$ for all $r, x, y \in R$

$[r, x] Rg(x) = 0$ for all $r, x \in R$…………………………………………………(6)

From the primeness of R, Equation (6) together with Brau's trick force that R is commutative or $g = 0$. So, for the case where R is commutative, Equation (1) becomes

$\big(F(x) - G(x)\big)x = 0$ for all $x \in R$, then $F = G$.

Otherwise , equation (4) becomes

$ryf(x)x - ryxG(x) + ryG(x)x = 0$ for all $r, x, y \in R$………………………….(7)

$f(x)x - xG(x) + G(x)x = 0$ for all $x \in R$…………………………………………..(8)

$f(x)x - F(x)x + G(x)x = 0$ for all $x \in R$…………………………………………..(9)

$\big(f(x) - F(x) + G(x)\big)y + \big(f(y) - F(y) + G(y)\big)x = 0$ for all $x, y \in R$…………(10)

The linearization of (9) gives

$\big(f(xt) - F(xt) + G(xt)\big)y + \big(f(y) - F(y) + G(y)\big)xt = 0$ for all $t, x, y \in R$……(11)

Right Multiplication of (10) by t gives

$\big(f(x) - F(x) + G(x)\big)yt + \big(f(y) - F(y) + G(y)\big)xt = 0$ for all $t, x, y \in R$…….(12)

Subtracting (12) from (11), we get

$\big(f(xt) - F(xt) + G(xt)\big)y - \big(f(x) - F(x) + G(x)\big)yt = 0$ for all $t, x, y \in R$ …….(13)

$\therefore \big(f(x) - F(x) + G(x)\big)ty - \big(f(x) - F(x) + G(x)\big)yt = 0$ for all $t, x, y \in R$ …….(14)

Replacing t by tr we get

$\therefore \big(f(x) - F(x) + G(x)\big)try - \big(f(x) - F(x) + G(x)\big)ytr = 0$ for all $r, t, x, y \in R$….(15)

Right multiplying (14) by r we get

$\therefore \big(f(x) - F(x) + G(x)\big)tyr - \big(f(x) - F(x) + G(x)\big)ytr = 0$ for all $r, t, x, y \in R$ …….(16)

Subtracting (16) from (15) we get

$\therefore \big(f(x) - F(x) + G(x)\big)try - \big(f(x) - F(x) + G(x)\big)tyr = 0$ for all $r, t, x, y \in R$

$\therefore \big(f(x) - F(x) + G(x)\big) (try - tyr) = 0$ for all $r, t, x, y \in R$

$\therefore \big(f(x) - F(x) + G(x)\big) t(ry - yr) = 0$ for all $r, t, x, y \in R$

$\therefore \big(f(x) - F(x) + G(x)\big) t[y, r] = 0$ for all $r, t, x, y \in R$ ……………………………….(17)

$\therefore$The primness of R together with (17) gives $f = F - G$

Now, we are to prove our main result.

Theorem 3.2: Let R be a 2-torsion free prime ring, J be a nonzero Jordan ideal of R and two generalized derivations F and G associated with f and g , respectively.

If $F(x)x - xG(x) = 0$ for all $x \in J$, then one of the following condition satisfied :

(1) R is commutative and $F = G$.

(2) G is a left multiplier and $F = G + f$.

Proof: Let R be a 2-torsion free prime ring, J be a nonzero Jordan ideal of R and two generalized derivations F and G associated with f and g , respectively.

Assume that

$F(x)x - xG(x) = 0$ for all $x \in J$ ...……………………………….(1)

Case (i) $Z(R) \cap J = \{0\}$

The linearization of (1) gives

$F(x)y + F(y)x - xG(y) + yG(x) = 0$ for all $x, y \in J$

Replacing x by $2x^2$ and y by $4yx^2$ in (1), we get

$F(2x^2)4yx^2 + F(4yx^2)2x^2 - 2x^2G(4yx^2) + 4yx^2G(2x^2) = 0$ for all $x, y \in J$

$yf(x^2)x^2 - x^2yg(x^2) - yx^2G(x^2) + yG(x^2)x^2 = 0$ for all $x, y \in J$………………(2)

Substituting $2[r, s]y$ in place of y in (2), where $r, s \in R$, we get

$$2[r, s]yf(x^2)x^2 - x^2 2[r, s]yg(x^2) - 2[r, s]yx^2G(x^2) + 2[r, s]yG(x^2)x^2 = 0$$

for all $r, s \in R , x, y \in J$

$$[r, s]yf(x^2)x^2 - x^2[r, s]yg(x^2) - [r, s]yx^2G(x^2) + [r, s]yG(x^2)x^2 = 0$$

for all $r, s \in R , x, y \in J$……………………………………………………………….(3)

$$[r,s]yf(x^2)x^2 - [r,s]x^2yg(x^2) - [r,s]yx^2G(x^2) + [r,s]yG(x^2)x^2 = 0$$

for all $r,s \in R$, $x,y \in J$……………………………………………………………………(4)

Subtracting (4) from (3), we get

$$[r,s]x^2yg(x^2) - x^2[r,s]yg(x^2) = 0$$

$$[[r,s],x^2]yg(x^2) = 0$$

$$\left[[r,s],x^2\right]yg\left(x^2\right) = 0$$

$\therefore [[r,s],x^2]Jg(x^2) = 0$ for all $r,s \in R$, $x \in J$………………………………………..(5)

By the primness of R together with $Lemma$ 2.2, we find $[[r,s],x^2] = 0$ or $g(x^2) = 0$. Clearly, in both cases, we arrive at $g(x^2) = 0$for all $x \in J$, This implies that $g = 0$ (by [5, $Lemma$ 3]). Now, replacing y by $2[r,uv]\, x$ in $yf(x)x - yxG(x) + yG(x)x = 0$ where $x,y \in J$ and $r \in R$, we get

$2[r,uv]\, xf(x)x - 2[r,uv]x\, xG(x) + 2[r,uv]\, xG(x)x = 0$ for all $x,y \in J$, $r \in R$

$[r,uv]\, f(x)x - [r,uv]\, xG(x) + [r,uv]\, G(x)x = 0$ for all $u,v,x \in J$, $r \in R$……….(6)

$\therefore [r,uv]\, f(x)x - [r,uv]\, F(x)x + [r,uv]\, G(x)x = 0$ for all $u,v,x \in J$, $r \in R$

$\therefore [r,uv](\, f(x)x - F(x)x + G(x)x\,) = 0$ for all $u,v,x \in J$, $r \in R$…………………..(7)

The fact that R is a non commutative prime ring forces that

$f(x)x - F(x)x + G(x)x = 0$for all $x \in J$ …………………………………………….(8)

The linearization of (8) gives

$f(x)y - F(x)y + G(x)y + f(y)x - F(y)x + G(y)x = 0$for all $x,y \in J$ …………..(9)

Replacing y by $2y\,[r,uv]$ in (9), we take, for all $u,v,x,y \in J$ and $r \in R$,

$f(x)2y\,[r,uv] - F(x)2y\,[r,uv] + G(x)2y\,[r,uv] + f(2y\,[r,uv])x - F(2y\,[r,uv])x + G(2y\,[r,uv])x = 0$for all $u,v,x,y \in J$, $r \in R$

$$f(x)y\,[r,uv] - F(x)y\,[r,uv] + G(x)y\,[r,uv]$$

$+f(y\,[r,uv])x - F(y\,[r,uv])x + G(y\,[r,uv])x = 0$for all $u,v,x,y \in J$, $r \in R$

$$f(x)y\,[r,uv] - F(x)y\,[r,uv] + G(x)y\,[r,uv]$$

$+f(y)[r,uv]x - F(y)[r,uv]x + G(y)[r,uv]x = 0$for all $u,v,x,y \in J$, $r \in R$ …..(10)

Right multiplying (9) by $[r,uv]$ we obtain, for all $u,v,x,y \in J$ and $r \in R$ we get

$$f(x)y[r,uv] - F(x)y[r,uv] + G(x)y[r,uv]$$

$+f(y)x[r,uv] - F(y)x[r,uv] + G(y)x[r,uv] = 0$for all $u,v\,x,y \in J$, $r \in R$ ……..(11)

Subtracting (11) from (10), we get,

$$\therefore f(y)[r,uv]x - F(y)[r,uv]x + G(y)[r,uv]x - (f(y)x[r,uv] - F(y)x[r,uv] + G(y)x[r,uv]) = 0$$

$$\therefore f(y)([r,uv]x - x[r,uv]) - F(y)([r,uv]x - x[r,uv]) + G(y)([r,uv]x - x[r,uv]) = 0$$

$$\therefore (\, f(y) - F(y) + G(y)\,)([r,uv]x - x[r,uv]) = 0$$

$\therefore (\, f(y) - F(y) + G(y)\,)[\,[r,uv],x] = 0$for all $u,v\,x,y \in J$, $r \in R$ ……………..(12)

Replacing x by $2x\,[s,t]$ where $s,t \in R$, we obtain

$\therefore (\, f(y) - F(y) + G(y)\,)J[\,[r,uv],[s,t]] = 0$for all $u,v\,,y \in J$, $r,s,t \in R$ ………..(13)

Since R is a non commutative prime ring, we get

$\therefore f(y) - F(y) + G(y) = 0$for all $y \in J$ ……………………………………………(14)

Replacing y by $4ry^2$in (14), where $r \in R$, we get

$\therefore f(4ry^2) - F(4ry^2) + G(4ry^2) = 0$for all $y \in J$, $r \in R$

$\therefore (\, f(r) - F(r) + G(r))\, y^2 = 0$for all $y \in J$, $r \in R$……………………………(15)

$\therefore f(r) - F(r) + G(r) = 0$for all $r \in R$

$$\therefore F = G + f$$

Case (ii) $Z(R) \cap J \neq \{0\}$

Let $0 \neq z \in Z(R) \cap J$ and replacing y by $2\,yz = y\,o\,z$ in $F(x)x = xG(x)$, we get

$yxf(z) = xyg(z)$ for all $x,y \in J$……………………………………………(16)

Replacing y by 2 $[r,s]$ y in (16), where $r,s \in R$, we get

$[r,s]yxf(z) = x[r,s]yg(z)$ for all $x,y \in J, r,s \in R$……………………………(17)

Left multiplication of (16) by $[r,s)]$ gives

$[r,s)]yxf(z) = [r,s)]xyg(z)$ for all $x,y \in J$, $r,s \in R$ …………………………………(18)

Subtracting (18) from (17), we get

$x[r,s]yg(z) - [r,s)]xyg(z) = 0$ for all $x,y \in J$, $r,s \in R$

$[\,[r,s]\,,x\,]yg(z) = 0$ for all $x,y \in J$, $r,s \in R$

$[\,[r,s]\,,x\,]Jg(z) = 0$ for all $x \in J$, $r,s \in R$ …………………(19)

Since R is a prime ring, Equation (19) forces that R is commutative of $g(z) = 0$.In this case where R is commutative we get, $F = G$. Otherwise, (16) forces that $f(z) = 0$. So replacing in (1) x by $2rz$, where $r \in R$, we get

$F(r)r = rG(r)$ for all $r \in R$ ……………………………………………(20)

Therefore, using Lemma 3.1 together with (20), we get the desired result.

As consequence of our main result we extend some results of [4] in more then, for any homomorphism of right R-modules $h : R \to R$ and any nonzero integerα, $\alpha F + h$ is a generalized derivation associated to the derivation αf. Applying this to Theorem 3.2, we get the following.

Corollary 3.3. Let R be a 2-torsion free prime ring, J be a nonzero Jordan ideal of R and two generalized derivations F and G associated with f and g, respectively. Then, for any homomorphism of right R-modules $h: R \to R$ and any nonzero integer α, if$F(x)x - \alpha x G(x) = hs(x)$ for all $x \in J$, then one of the following holds.

(1) R is commutative and $F = \alpha G + h$.

(2) αG is a left multiplier and $F = \alpha G + h + f$.

For instance if we take (in Corollary 3.3) $h = \beta id_R$ (where id_R is the identity map on R and β is an integer), then we get the following result:

Corollary 3.4. Let R be a 2-torsion free prime ring, J be a nonzero Jordan ideal of R and two generalized derivations F and G associated with f and g, respectively. Then, for any two integers $\alpha \neq 0$ and β, if $F(x)x - \alpha x G(x) = \beta x^2$ for all $x \in J$, then one of the following holds:

(1) R is commutative and $F = \alpha G + \beta id_R$.

(2) αG is a left multiplier and $F = \alpha G + \beta id_R + f$.

Now we give the first desired result which is a generalization of [4, Theorem 3.7]

Corollary 3.5: Let R be a 2-torsion free prime ring, J be a nonzero Jordan ideal of R and two generalized derivations F and G of R associated with $f \neq 0$ and g, respectively, such that $G(x^2) = 2xF(x)$ for all $x \in J$, then R is commutative and $2F = G + g$.

Proof. By hypothesis,

$G(x^2) + xg(x) = 2xF(x)$for all $x \in J$

Then $G(x)x - 2xF(x) = -xg(x)$ for all $x \in J$

Therefore, the result using Corollary 3.3

Corollary 3.6. Let R be a 2-torsion free prime ring, J be a nonzero Jordan ideal of R and two generalized derivations F and G of R associated with $f \neq 0$ and $g \neq 0$, respectively,, respectively such that$F(u^2) - 2uF(u) = G(u^2) - 2uG(u)$for all $u \in J$, then R is commutative and $F - G = f - g$.

Proof. By hypothesis,

$F(u^2) - 2uF(u) = G(u^2) - 2uG(u)$ for all $u \in J$

Since F and G are additive maps, above equation can be rewritten as follows.

$(F - G)(u^2) = 2u(F - G)(u)$ for all $u \in J$

If we set $K = F - G$, we get $K(u^2) = 2u\,K(u)$ for all $u \in J$. Then by Corollary 3.5, we obtain the result.

Now our aim to give a generalization of [4, Theorem 3.6]. As done before we prefer at first giving the following general result.

Also, as before, if we consider a generalized derivation F associated to a derivation, f, then, for any homomorphism of left R-modules $h: R \to R$and any nonzero integer$\alpha, \alpha F + h$ is a generalized derivation associated to the derivation αf.Applying this to Theorem 3.2, we get the following .

Corollary 3.7. Let R be a 2-torsion free prime ring, J be a nonzero Jordan ideal of R and two generalized derivations F and G associated with f and g, respectively. Then, for any homomorphism of left R-modules $h: R \to R$ and any nonzero integer α, if $F(x)x - \alpha x G(x) = h(x)x \; for \; all \; x \in J$, then one of the following holds:

(1) R is commutative and $F - h = \alpha G$.

(2) αG is s left multiplier and $F - h = \alpha G + f$.

As a consequence we get the following generalization of [4, Theorem 3.6].

Corollary 3.8. Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If there are generalized derivations F and G of R associated with derivations f and$g \neq 0$, respectively, such that $G(x^2) = 2F(x)x \; for \; all \; x \in J$, then R is commutative and $2F = G + g$.

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