Flow in Constricted Tube of Varying Cross Section with Permeable Wall

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Abstract: ***In this paper, we study the low Reynolds number steady flow in constricted tube of varying cross section with permeable wall. The fluid is assumed to be incompressible and Newtonian. The wall assumed to be rigid and permeable. The wall permeability is assumed to be a function of axial distance and obeys Starling's Law. We are interested to analyze the effects of Reynolds number and permeability on flow characteristics when the initial flux in the tube is prescribed. The effect of variable permeability of the wall on various parameter on flow characteristics is discussed.***

Keywords: Numerical Solution of Differential Equation, Fluid Mechanics, Reynolds number

1. Introduction

Flow in tubes of varying cross-section is a good area for research work due to its importance in physiological and engineering flow problems. In particular, it plays a significant role in understanding the flow in blood vessels. Most of these studies have considered the tube walls to be impermeable. Flow through tube of uniform cross section with permeable wall has been investigated due to its application in engineering flow problem . Berman(1953) [2] worked on flow through ducts with permeable wall as suction/injection problem where normal velocity of the fluid at the wall is prescribed and these studies suction/injection velocity prescribed at the wall is constant. Macey(1965)[11] prescribed flux as an exponentially decreasing function of axial distance to account for the fluid absorption of the wall. Frielman and Gill(1967)[6] have studied flow though cylindrical tube with permeable walls with reference to flow in the proximal renal tubes.. Manton(1971)[12] have studied for pulsatile flow for tubes of slowly varying cross section. Apelblat , Karzin-katchesky and Silberbarg (1974)[1] presented Mathematical analysis for the fluid exchange across the capillary wall using Sterling law. Quaile and Levy (1975)[16] investigated flow through ducts with permeable wall as suction / injection problem in these studies constant suction /injection velocity prescribed at the wall. Varma and Sachati (1975)[20] investigated flow of a power law fluid through circular tube with porous material by property defining the non slip conditions.

Radhakrishnamachrya (1978)[17] studied flow of a dusty fluid in constricted channel. Bestman (1981)[3] analyzed pulsatile flow of a Rivlin- Ericksen fluid at low Reynolds number non Newtonian flow in slowly varying cross section at asymmetrical tubes. Also Radhakrishnamachrya and Peeyush Chandra and Kaimel (1981)[18] the Hydrodynamical problem of flow in proximal renal tubule is investigated by considering axisymmetric flow of a viscous , incompressible fluid though long narrow tube of varying cross section with reabsorption at the wall. Chandra, Peeyush and Radhakrishnamachrya (1983)[4] worked on fluid exchange across converging/diverging tube walls. Colgan and Terril (1989)[5] presented first order solution for asymmetric flow through circular pipe of slowly varying cross section valid for arbitrary Reynolds number. Krishna Prasad and Peeyush Chandra (1990)[8] have worked on the low Reynolds number flow of a viscous incompressible fluid in channels of slowly varying cross-section with permeable boundaries has been studied. The effect of various parameters on the flow characteristics like wall shear stress, pressure drop and volumetric flow rate has been discussed. Krishna Prasad and Peeyush Chandra (1992)[9] have studied low Reynolds number flow of viscous incompressible Newtonian fluid in cylindrical tube of varying cross section with absorbing walls. Krishna Prasad and Peeyush Chandra have(1992)[10]have studied Pulsatile flow in circular tubes varying cross section with permeable wall .Sarin(1997) [19] fully developed steady laminar flow of an idealized elastic-viscous liquid through a curve tube with elliptic cross section. M.Zakaria (2002)[13]worked on the equation of a polar fluid of hydromantic fluctuating through a porous medium. M.A.A.Mahmoud and M.A.E.Mohmoud (2005) [14] have studied the boundary layer flow of power-law non Newtonian fluid over continuously moving surface in presence of a magnetic field. H.Beirao da Veiga (2008) [7] have studied the motion of non Newtonian fluid with shear dependent viscosity between two cylinders. Mario, Dannis and Amaru Gonzalez(2017) [15] have worked on eleasto -viscoplastic fluid in tubes of varying cross section. The fluid exchange across the wall is accounted for prescribing the normal velocity of the fluid at the wall. A perturbation analysis has been carried out for flow Reynolds number flows and for small amplitude of oscillation.

We consider steady flow of an incompressible fluid in a rigid tube of slowly varying cross-section with absorbing wall. The effect of fluid absorption through permeable wall is accounted by prescribing flux as an arbitrary function of axial distance .The fluid exchange across the tube wall is accounted either by prescribing normal fluid velocity at the wall which is equivalent to prescribing flow flux at different cross-sections of the tube or through Starling's law which states that normal velocity of the fluid at the wall is proportional to the pressure difference across the vessel wall. In this paper, we study low Reynolds number flow in constricted tubes of varying cross section with permeable wall. Further, we assume that wall permeability K is a function of axial distance. An initial value problem is formulated where flux and mean pressure at the initial cross section have been prescribed. We are interested to study the effects of Re and K on flow characteristics.

2. Formulation of the Problem

Consider steady flow of a Newtonian incompressible fluid in an axisymmetric tube of varying cross-section with permeable wall. Using cylindrical polar coordinates (X, R, θ) where R = O is the axis of symmetry for the tube, the equations of motion and continuity are given as:

$$UU_X + V U_R = -\frac{1}{\rho}P_X + \nu\left[U_{XX} + \frac{(RU_R)_R}{R}\right] \ldots\ldots\ldots(1)$$

$$UU_X + V V_R = -\frac{1}{\rho}P_X + \nu\left[V_{XX} + \frac{1}{R}(RV_R)_R - \frac{V}{R^2}\right] \ldots\ldots\ldots(2)$$

$$U_X + \frac{(RV)_R}{R} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where (U, V) are the fluid velocity components in (X, R) directions respectively, P is the pressure, ν is the kinematic coefficient of viscosity and ρ is the constant fluid density.

We consider tube of slowly varying cross-section, and hence, the radius of the tube

R = a (X) is given as:

$a(x) = S(\varepsilon X/a_0)$

$\varepsilon = a_0/L \ll 1$,

$$S(0)=1 \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Where ε is the wall variation parameter, a_0 is the tube radius at the initial cross-section, L is the characteristic length and $S(\varepsilon X/a_0)$ is an arbitrary function of X.

The fluid exchange across the permeable wall is given by Starling's law and the net external pressure acting on the surface of the wall is assumed to be constant. This gives the normal fluid velocity at the tube wall as:

$$V - a_x U = K(P - P_{ext}) \text{ at } R = a(x) \quad \ldots\ldots\ldots\ldots(5)$$

The tangential velocity of the fluid at the wall is zero, hence,

$$U + a_x V = 0 \text{ at } R=a(x) \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The asymmetry of the flow implies

$$U_R = 0, V = 0 \text{ at } R=0 \ldots\ldots\ldots\ldots\ldots(7)$$

Further, we prescribe the mean pressure P_{mean} i. e.

$$P_{mean} = \frac{1}{\pi a^2(x)}\int_0^{a(x)} 2\pi R\, P\, dR \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$\text{And the flux Q, } Q = \int_0^{a(x)} 2\pi R\, U\, dR \ldots\ldots\ldots\ldots(9)$$

At the initial cross-section (X = 0) as P_{in} and Q_0 respectively, which gives

$$P_{mean} = P_{in}$$
$$Q = Q_0 \text{ at } X=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

The wall permeability is assumed to be a function of axial distance **K(X)=mk(1+nkX)**

Where mk and nk are real constants less than 1. It may be noted that when n=0, our case reduces to constant permeability as given by [9] and [10].

Analysis and Method of Solution:

Using the non-dimensional quantities,

$x = \varepsilon X/a_0$, $r = \frac{R}{a_0}$, $u = 2\pi a_0^2 U/Q_0$,

$v = 2\pi a_0^2 V/\varepsilon Q_0$, $(p, p_{ext}) = 2\pi a_0^2 \varepsilon (P, P_{ex})/\nu\rho Q_0$,

$k = \nu\rho K/\varepsilon^2 a_0$, $q = Q/Q_0$

and the perturbation technique in terms of parameter ε with

$(u, v, p, q) = (u^{(0)}, v^{(0)}, P^{(0)}, q^{(0)}) + \varepsilon(u^{(1)}, v^{(1)}, P^{(1)}, q^{(1)}) + o(\varepsilon^2)$,

We get Zeroth order velocity components as follows

$$u^{(0)} = \frac{1}{4}p_x^{(0)}(r^2 - s^2) \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

$$V^{(0)} = -\frac{1}{16}r\left[p_{xx}^{(0)}(r^2 - 2s^2) - 4S\, S_x p_x^{(0)}\right] \ldots\ldots(12)$$

First order velocity components as follows

$$u^{(1)} = \frac{1}{4}p_x^{(1)}(r^2 - s^2) + \frac{Re}{2304}p_x^{(0)}\left[p_{xx}^{(0)}(2r^6 - 9r^4s^2 - 36r^2s^4 - 29s^6)\right.$$
$$\left. - 72s^3 S_x p_x^{(0)}(r^2 - s^2)\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$V^{(1)} = -\frac{1}{16}r\left[p_{xx}^{(1)}(r^2 - 2s^2) - 4S\, S_x p_x^{(1)}\right]$$
$$-\frac{Re}{9216}rp_x^{(0)}\left[p_{xx}^{(0)2} + p_x^{(0)}p_{xxx}^{(0)}(r^6 - 6r^4s^2 + 36r^2s^4 - 58s^6)\right.$$
$$+12\, S\, S_x p_x^{(0)} p_{xx}^{(0)}(24r^2s^2 - r^4 - 41s^4)$$
$$\left. + 72s^{(2)}p_x^{(0)^2}\{S_{xx}(r^2 - 2s^2) + S_x^2(3r^2 - 10s^2)\right) \ldots (14)$$

Flow Rate and Wall Shear Stress

The non-dimensional volumetric flow rate (q) Wall shear stress is given by:

$$q = \int_0^{S(x)} ru\, dr$$

$$q = -\frac{s^4}{16}\left[p_x^{(0)} + \varepsilon\left\{p_x^{(1)} + \frac{R_e}{16}s^3 p_x^{(0)}\left(12k(p^{(0)} - p_{ext}) - S_x p_x^{(0)}\right)\right\}\right] + o(\varepsilon^2) \quad \ldots.(15)$$

The wall shear stress in non-dimensional form is given as :

$$T_w = 2\pi a_0^3 \tau_w/\rho\nu Q_0 \ldots\ldots\ldots\ldots\ldots(16)$$

$$T_w = \frac{S}{2}p_x^{(0)} + \varepsilon\left[p_x^{(1)} + \frac{R_e}{24}Sp_x^{(0)}\left\{16k(p^{(0)} - p_{ext})\right.\right.$$
$$\left.\left. - S^2 S_x p_x^{(0)}\right\}\right] + o(\varepsilon^2) \ldots\ldots\ldots (17)$$

Calculation of Pressure

Here, the expressions for various flow variables are given in terms of $p^{(0)}, p^{(1)}$ and their derivatives. These flow variables can be determined once $p^{(0)}$ and $p^{(1)}$ are evaluated. The equation governing pressure is obtained through Starling's law.

Thus, using conditions expression for $V^{(0)}$ and $V^{(1)}$, we get the following differential equations for $p^{(0)}$ and $p^{(1)}$,

$$p_{xx}^{(0)} + 4\frac{S_x}{s}p_x^{(0)} - 16\frac{k}{s^3}(p^{(0)} - p_{ext}) = 0 \ldots\ldots.(18)$$

$$p_{xx}^{(1)} + 4\frac{S_x}{s}p_x^{(1)} - 16\frac{k}{s^3}p^{(1)} = -\frac{R_e}{64}S^2\left[3S^2(p_x^{(0)^2} + p_x^{(0)}p_{xxx}^{(0)})\right.$$
$$\left. + 40S\, S_x p_x^{(0)} p_{xx}^{(0)} + 8p_x^{(0)^2}(sS_{xx} + 7S_x^2)\right] \ldots\ldots\ldots(19)$$

$$p^{(0)} = p_{in},\ p_x^{(0)} = -16 \ldots\ldots\ldots\ldots\ldots\ldots(20)$$

$$p^{(1)} = 0\,,\, p_x^{(1)} = 4\,R_e[3k\,(p^{(0)} - p_{ext}) + 4s_x] \ldots\ldots(21)$$

The differential eqns. (21) and (22) with initial conditions form two point initial value problems for $p^{(0)}$ and $p^{(1)}$ for a given tube geometry, these equations can be solved and the mean pressure drop ΔP at a given cross-section

$$\Delta p = p_{mean}^{(0)} - p_{mean}^{(1)} = p_{in} - p^{(0)}(x) - \varepsilon p^{(1)}(x) + O(\varepsilon^2) \ldots (22)$$

can be calculated.

3. Numerical Solution and Discussion

In general, analytical solutions of the equations (18), (19) are not feasible and equations have to be solved numerically for a given S(x). However, in a particular case of S(x) =(2-exp(-(x-0.5)*(x-0.5)))/(2-exp(-0.25)) constricted tube .It is possible to find analytic solution for $p^{(0)}$ analytically. But in this case also, it becomes very tedious to solve for $p^{(0)}$ analytically. In view of this, fourth order R-K Method is used to evaluate $p^{(0)}$ and $p^{(1)}$numerically. Hence, we evaluate the expressions flow rate (Q) and wall shear stress $|T_W|$.

We have taken $\varepsilon = 0.1$ and 0.05 in fig.1 ,fig.3 , fig.5 and fig.7 variation of flow rate Q has been shown. The effect of Re and permeability K on flow rate (Q)have been shown in constricted .The flow flux decreases for this tube .The effect of increase in permeability is to decrease the flux.

In fig.2, fig.4 fig.6 , fig.8 Variation of wall shear stress $|T_W|$ has been shown. The maximum value of wall shear stress is observed around the point of constriction. when permeability increases the wall shear stress decreases.

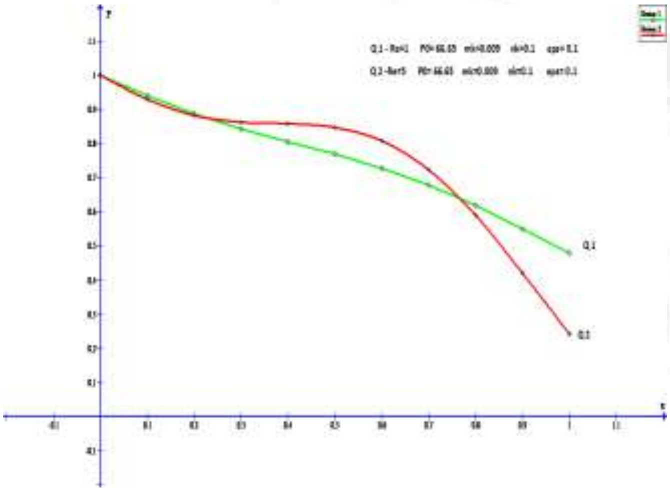

Figure 1: Flow rate Q vs axial distance X for constricted tube for Re=1,Re=5,mk=0.009 nk=0.1 eps=0.1

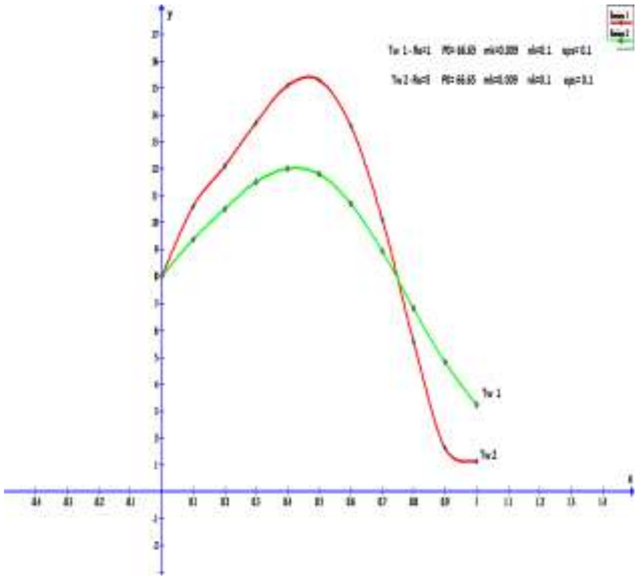

Figure 2: Wall shear stress Tw Vs axial X distance for constricted tube for Re=1 , Re=5, mk = 0.009 nk=0.1 eps=0.1

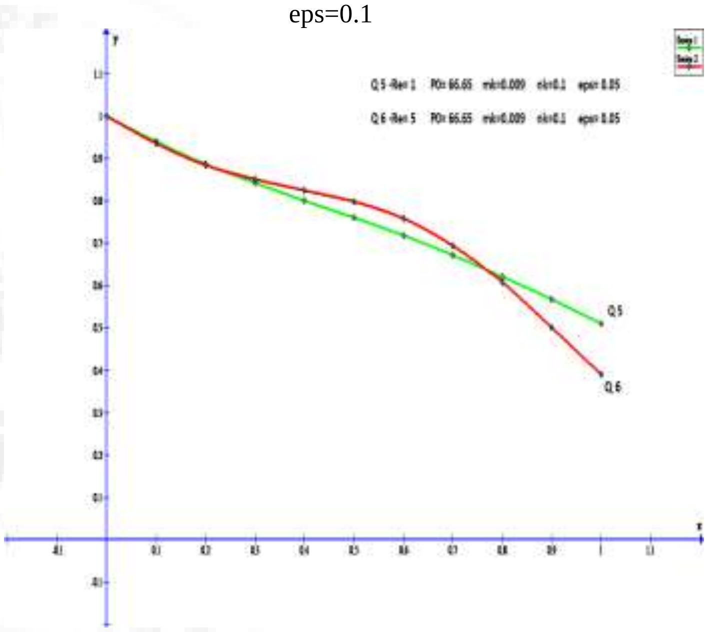

Figure 3: Flow rate Q vs axial distance X for constricted tube for Re=1 , Re=5, mk = 0.009 nk=0.1 eps=0.05

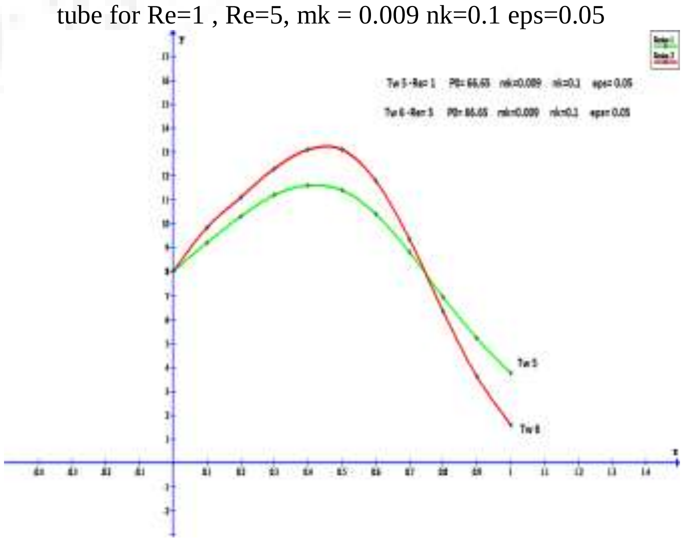

Figure 4: Wall shear stress Tw Vs axial X distance for constricted tube for Re=1, Re=5, mk = 0.009 nk=0.1 eps=0.05

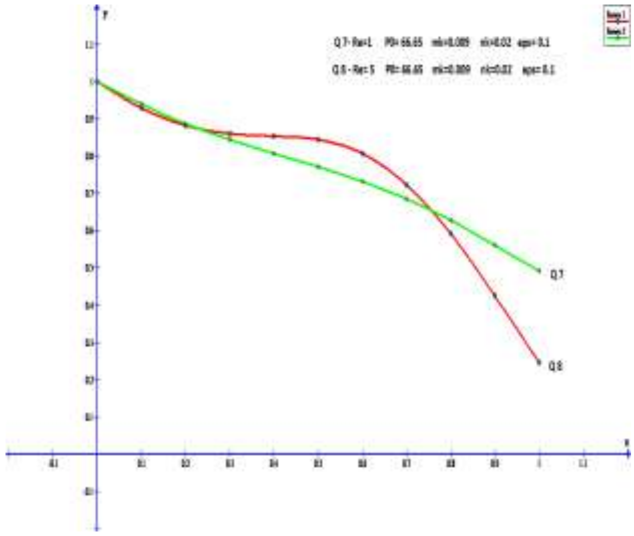

Figure 5: Flow rate Q vs axial distance X for constricted tube for Re=1 , Re=5, mk = 0.009 nk= 0.02 eps=0.1

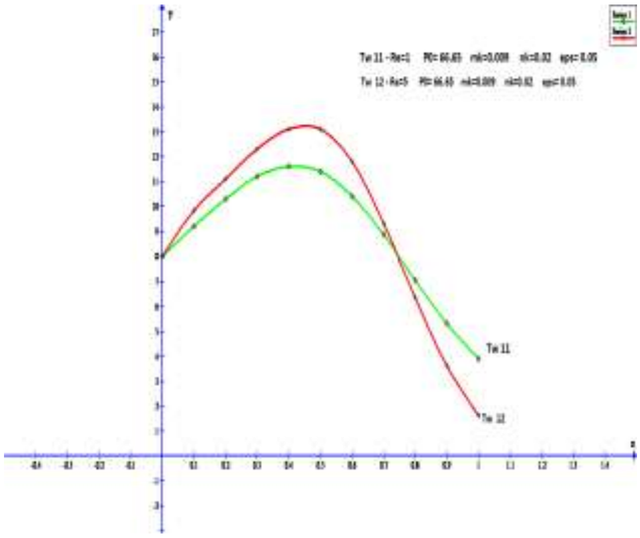

Figure 8: Wall shear stress Tw Vs axial X distance for constricted tube for Re=1, Re=5,mk = 0.009 nk=0.02 eps=0.05

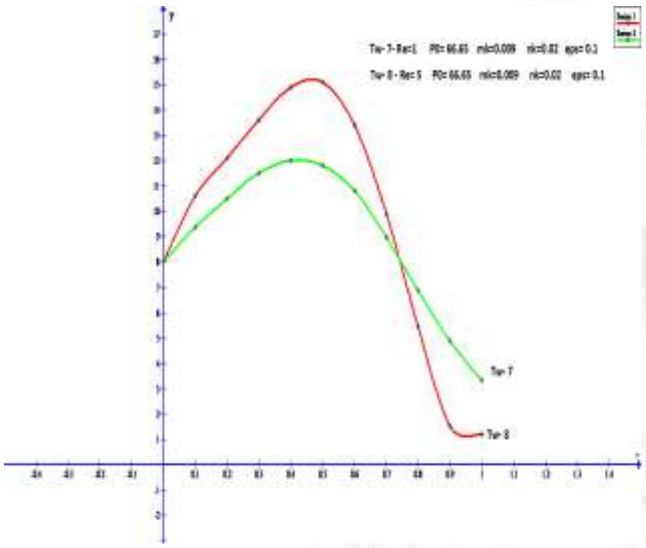

Figure 6: Wall shear stress Tw Vs axial X distance for constricted tube for Re=1, Re=5, mk = 0.009 nk=0.02 eps=0.1

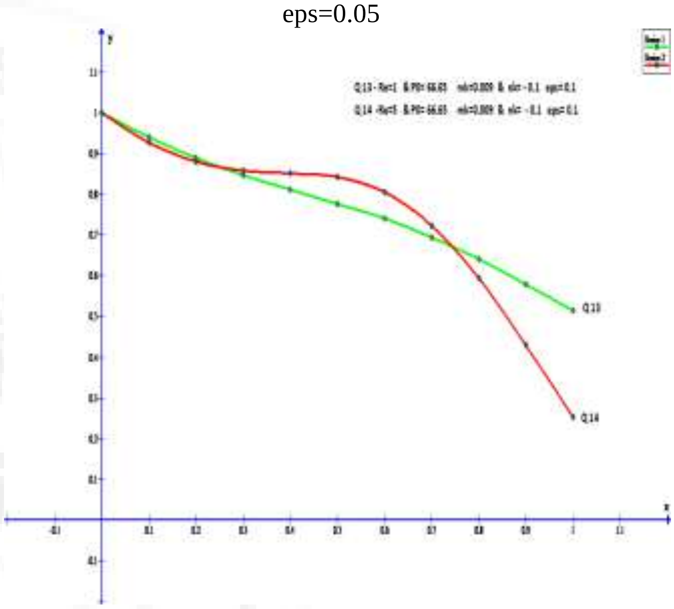

Figure 9: Flow rate Q vs axial distance X for constricted tube for Re=1 , Re=5,mMk = 0.009 nk= - 0.1 eps=0.1

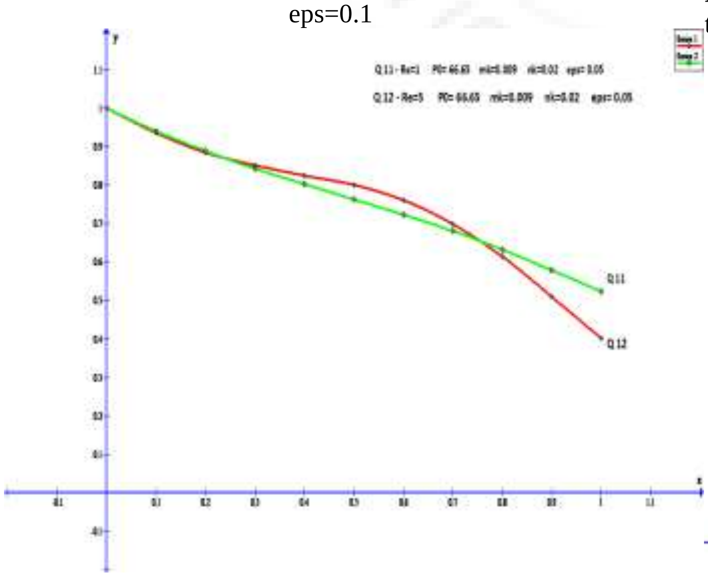

Figure 7: Flow rate Q vs axial distance X for constricted tube for Re=1 , Re=5, mk = 0.009 nk= 0.02 eps=0.05

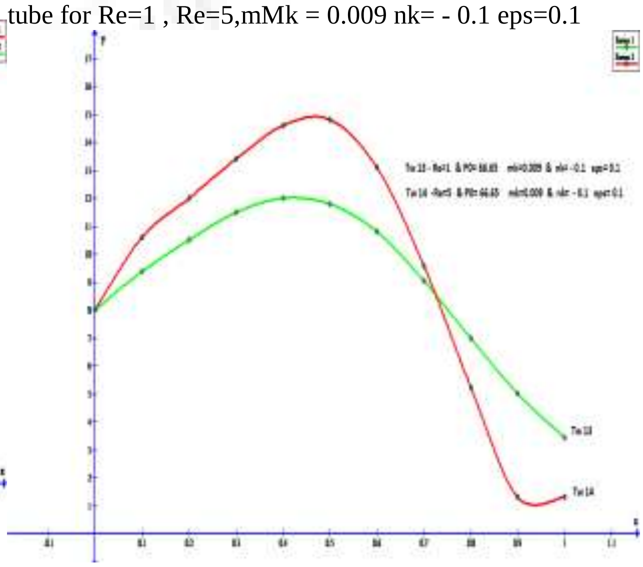

Figure 10: Wall shear stress Tw Vs axial X distance for constricted tube Re=1, Re=5,mk = 0.009 nk= - 0.1 eps=0.1

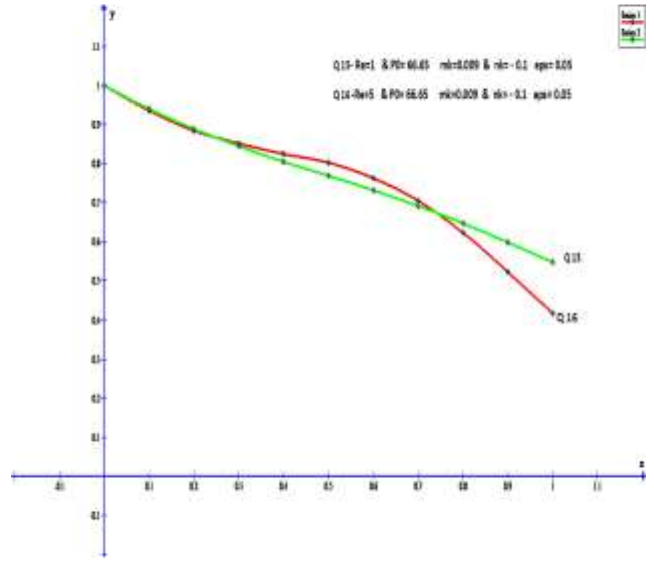

Figure 11: Flow rate Q vs axial distance X for constricted tube for Re=1, Re=5, mk = 0.009 nk= - 0.1 eps=0.05

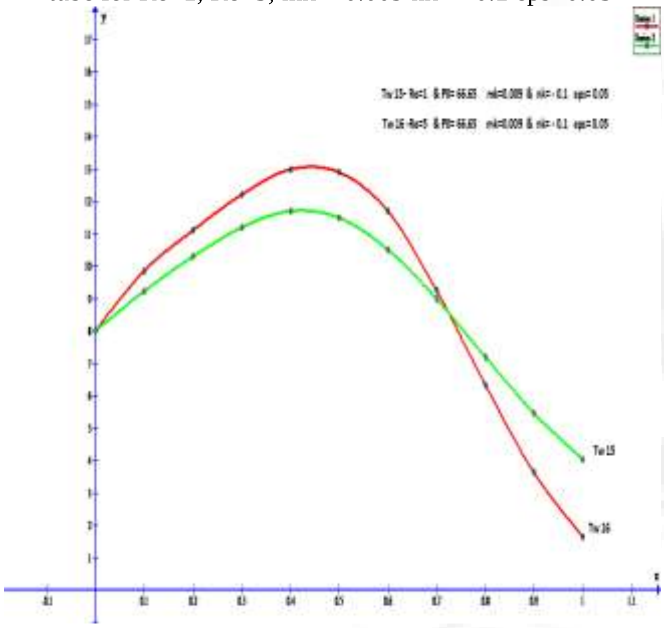

Figure 12: Wall shear stress Tw Vs axial X distance for constricted tube Re=1, Re=5,mk = 0.009 nk= - 0.1 eps=0.05

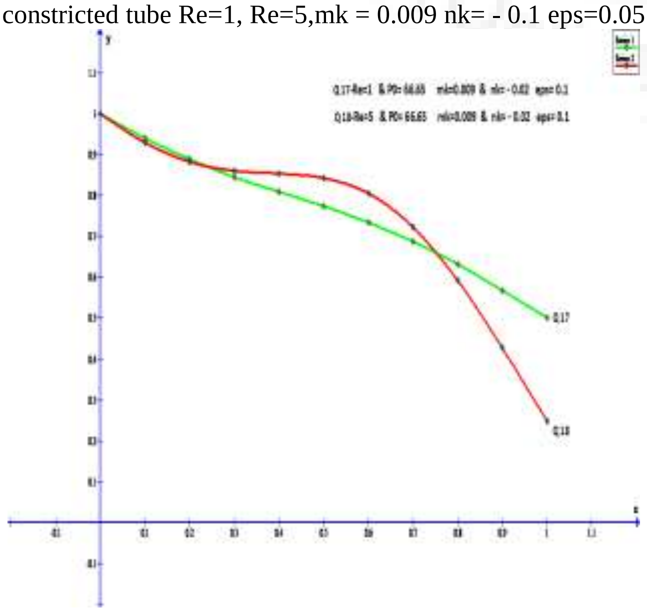

Figure 13: Flow rate Q vs axial distance X for constricted tube for Re=1 , Re=5, mk = 0.009 nk= - 0.02 eps=0.1

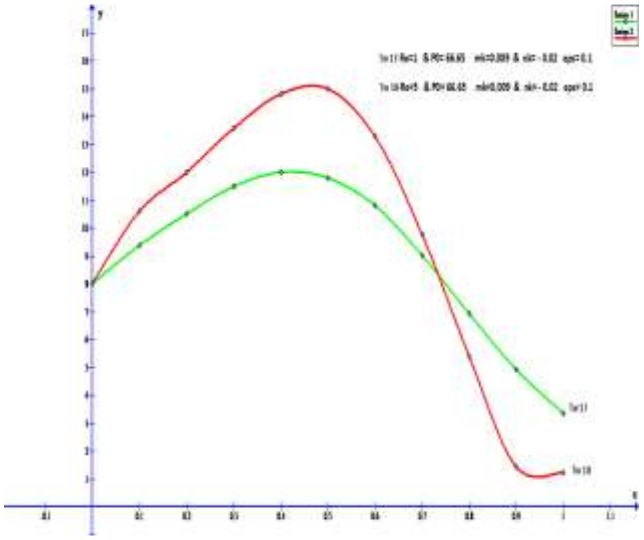

Figure 14: Wall shear stress Tw Vs axial X distance for constricted tube Re=1, Re=5,m k = 0.009 nk= - 0.02 eps=0.1

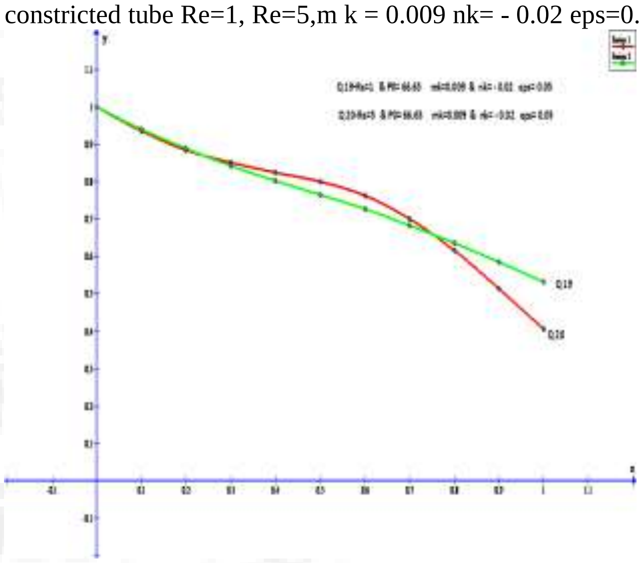

Figure 15: Flow rate Q vs axial distance X for constricted tube for Re=1 , Re=5, mk = 0.009 nk= - 0.02 eps=0.05

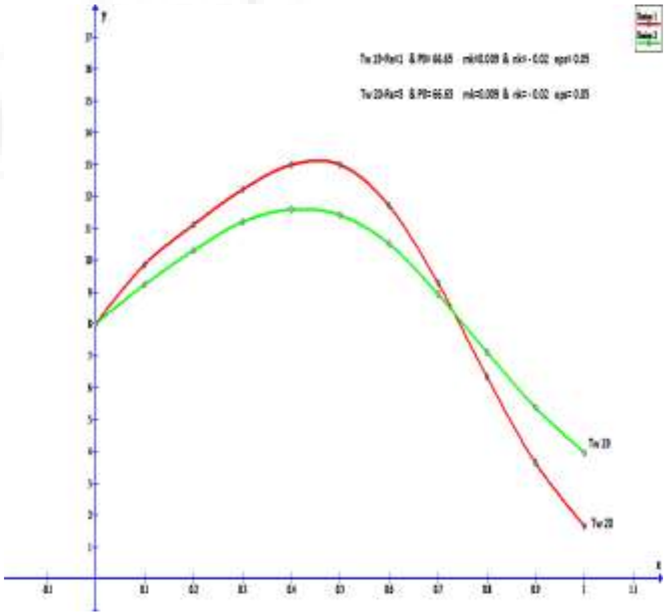

Figure 16: Wall shear stress Tw Vs axial X distance for constricted tube Re=1, Re=5, mk = 0.009, nk= - 0.02, eps=0.05

4. Conclusion

Using numerical values of $P^{(0)}$ and $P^{(1)}$ and their derivatives, value of flow rate (Q) and wall shear stress (I Tw I) are calculated .We have taken $\epsilon = 0.05$ and $\epsilon = 0.1$ for numerical calculation. The numerical solution obtain by fourth order using R-k method.

In this paper, we have considered effect of wall permeability (Kp) and Reynolds number Re on wall shear stress, pressure and flow flux for constricted tube. It is observed all these flow value of flow rate decreases as the wall permeability increases. In this tube maximum value of wall shear stress (I Tw I) is observed around the point of contraction. Also as Re increases wall shear stress increases in the constricted region of the tube then decreases in diverging region of tube.

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